

HILBERT COEFFICIENTS AND SEQUENTIALLY COHEN-MACAULAY MODULES

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ABSTRACT. The purpose of this paper is to present a characterization of sequentially Cohen-Macaulay modules in terms of its Hilbert coefficients with respect to distinguished parameter ideals. The formulas involve arithmetic degrees. Among corollaries of the main result we obtain a short proof of Vasconcelos Vanishing Conjecture for modules and an upper bound for the first Hilbert coefficient.

1. INTRODUCTION

Let (R, \mathfrak{m}) be a Noetherian local ring with the maximal ideal \mathfrak{m} and I an \mathfrak{m} -primary ideal of R . Let M be a finitely generated R -module of dimension d . It is well known that there exists a polynomial $p_I(n)$ of degree d with rational coefficients, called the Hilbert-Samuel polynomial, such that $\ell(M/I^{n+1}M) = p_I(n)$ for all large enough n . Then, there are integers $e_i(I, M)$ such that

$$p_I(n) = \sum_{i=0}^d (-1)^i e_i(I, M) \binom{n+d-i}{d-i}.$$

These integers $e_i(I, M)$ are called the Hilbert coefficients of M with respect to I . In particular, the leading coefficient $e_0(I, M)$ is called the multiplicity of M with respect to I and $e_1(I, M)$ is called by Vasconcelos the Chern number of I with respect to M . Although the theory of multiplicity has been rapidly developing for the last 50 years and proved to be a very important tool in algebraic geometry and commutative algebra, not so much is known about the Hilbert coefficients $e_i(I, M)$ with $i > 0$. At the conference in Yokohama 2008, W. V. Vasconcelos [V2] posed the following conjecture:

The Vanishing Conjecture: Assume that R is an unmixed, that is $\dim(\hat{R}/P) = \dim R$ for all $P \in \text{Ass } \hat{R}$, where \hat{R} is the \mathfrak{m} -adic completion of R . Then R is a Cohen-Macaulay local ring if and only if $e_1(\mathfrak{q}, R) = 0$ for some parameter ideal \mathfrak{q} of R .

Recently, this conjecture has been settled by L. Ghezzi, J.-Y. Hong, K. Ozeki, T. T. Phuong, W. V. Vasconcelos and the second author in [GGHOPV]. Moreover, the second author showed in [G] how one can use Hilbert coefficients of parameter ideals to study

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many classes of non-unmixed modules such as Buchsbaum modules, generalized Cohen-Macaulay modules, Vasconcelos modules.... The aim of our paper is to continue this research direction. Concretely, we will give characterizations of a sequentially Cohen-Macaulay module in term of its Hilbert coefficients with respect to certain parameter ideals (Theorem 4.5). Recall that sequentially Cohen-Macaulay module was introduced first by Stanley [St] for graded case. In the local case, a module M is said to be a sequentially Cohen-Macaulay module if there exists a filtrations of submodules $M = D_0 \supset D_1 \supset \dots \supset D_s$ such that $\dim D_i > \dim D_{i+1}$ and D_i/D_{i+1} are Cohen-Macaulay for all $i = 0, 1, \dots, s-1$ (see [Sc], [CN]). Then M is a Cohen-Macaulay module if and only if M is an unmixed sequentially Cohen-Macaulay module. Therefore, as an immediate consequence of our main result, we get again the answer to Vasconcelos' Conjecture for modules. Furthermore, Theorem 4.5 let us to get several interesting properties of the Chern numbers of parameter ideals on non-unmixed modules. Especially, we can prove a slight stronger than Theorem 3.5 of M. Mandal, B. Singh and J. K. Verma in [MSV] about the non-negativity of the Chern number of any parameter ideal with respect to arbitrary finitely generated module (Corollary 4.7).

This paper is divided into 4 sections. In the next section we recall the notions of dimension filtration, good parameter ideals and distinguished parameter ideals following [Sc], [CN], [CC1], [CC2], and prove some preliminary results on the dimension filtration. We discuss in Section 3 the relationship between Hilbert coefficients and arithmetic degrees (see [BM], [V]) of an \mathfrak{m} -primary ideal. The last section is devoted to prove the main result and its consequences.

2. THE DIMENSION FILTRATION

Throughout this paper, (R, \mathfrak{m}) is a Noetherian local ring and M is a finitely generated R -module of dimension d .

Definition 2.1. ([CC1], [CC2], [CN]) A filtration $\mathcal{D} : M = D_0 \supset D_1 \supset \dots \supset D_s = H_{\mathfrak{m}}^0(M)$ of submodules of M is said to be a *dimension filtration*, if D_i is the largest submodule of D_{i-1} with $\dim D_i < \dim D_{i-1}$ for all $i = 1, \dots, s$. A system of parameters $\underline{x} = x_1, \dots, x_d$ of M is called a *good system of parameters* of M , if $N \cap (x_{\dim N+1}, \dots, x_d)M = 0$ for all submodules N of M with $\dim N < d$. A parameter ideal \mathfrak{q} of M is called a *good parameter ideal*, if there exists a good system of parameters $\underline{x} = x_1, \dots, x_d$ such that $\mathfrak{q} = (\underline{x})$.

Now let us briefly give some facts on the dimension filtration and good systems of parameters (see [CC1], [CC2], [CN]). Let \mathbb{N} be the set of all positive integers. We denote by

$$\Lambda(M) = \{r \in \mathbb{N} \mid \text{there is a submodule } N \text{ of } M \text{ such that } \dim N = r\}.$$

Because of the Noetherian property of M , the dimension filtration of M and $\Lambda(M)$ exist uniquely. Therefore, throughout this paper we always denote by

$$\mathcal{D} : M = D_0 \supset D_1 \supset \dots \supset D_s = H_{\mathfrak{m}}^0(M)$$

the dimension filtration of M with $\dim D_i = d_i$, and $\mathcal{D}_i = D_i/D_{i+1}$ for all $i = 0, \dots, s-1$. Then we can check that

$$\Lambda(M) = \{d_i = \dim D_i \mid i = 0, \dots, s-1\}.$$

In this case, we also say that the dimension filtration \mathcal{D} of M has the length s . Moreover, let $\bigcap_{\mathfrak{p} \in \text{Ass } M} N(\mathfrak{p}) = 0$ be a reduced primary decomposition of submodule 0 of M , then $D_i = \bigcap_{\dim(R/\mathfrak{p}) \geq d_{i-1}} N(\mathfrak{p})$. Especially, if we set $\text{Assh}(M) = \{\mathfrak{p} \in \text{Ass}(M) \mid \dim R/\mathfrak{p} = \dim M\}$, then the submodule

$$D_1 = \bigcap_{\mathfrak{p} \in \text{Assh}(M)} N(\mathfrak{p})$$

is called the unmixed component of M and denoted by $U_M(0)$. It should be mentioned that $U_M(0)$ is just the largest submodule of M having the dimension strictly smaller than d . Moreover, $H_{\mathfrak{m}}^0(M) \subseteq U_M(0)$ and $H_{\mathfrak{m}}^0(M) = U_M(0)$ if $\text{Ass}(M) \subseteq \text{Assh}(M) \cup \{\mathfrak{m}\}$. Put $N_i = \bigcap_{\dim(R/\mathfrak{p}) \leq d_i} N(\mathfrak{p})$. Therefore $D_i \cap N_i = 0$ and $\dim(M/N_i) = d_i$. By the Prime Avoidance Theorem, there exists a system of parameters $\underline{x} = (x_1, \dots, x_d)$ such that $x_{d_i+1}, \dots, x_d \in \text{Ann}(M/N_i)$. It follows that $D_i \cap (x_{d_i+1}, \dots, x_d)M \subseteq N_i \cap D_i = 0$ for all $i = 1, \dots, s$. Thus by the definition of the dimension filtration, $\underline{x} = x_1, \dots, x_d$ is a good system of parameters of M , and therefore the set of good systems of parameters of M is always non-empty. Let $\underline{x} = x_1, \dots, x_d$ be a good system of parameters of M . It easy to see that x_1, \dots, x_{d_i} is a good system of parameters of D_i , so is $x_1^{n_1}, \dots, x_{d_i}^{n_{d_i}}$ for any d -tuple of positive integers n_1, \dots, n_d . With notations as above we have

Lemma 2.2. *Let $\mathcal{F} : M = M_0 \supset M_1 \supset \dots \supset M_t$ be a filtration of submodules of M . Then the following statements are equivalent:*

- (1) $\dim M_t \leq 0$, $\dim M_{i+1} < \dim M_i$, and $\text{Ass}(M_i/M_{i+1}) \subseteq \text{Assh}(M_i/M_{i+1}) \cup \{\mathfrak{m}\}$ for all $i = 0, 1, \dots, t-1$.
- (2) $s = t$ and D_i/M_i has a finite length for each $i = 1, \dots, s$.

When this is the case, we have $\dim M_i = d_i$.

Proof. (2) \Rightarrow (1) is trivial from the definition of the dimension filtration.

(1) \Rightarrow (2). We show recursively on i that D_i/M_i has a finite length for all $i \leq t$. In the case $i = 1$, we have $M_1 \subseteq D_1$ and so that

$$\text{Ass}(D_1/M_1) \subseteq \text{Ass}(M/M_1) \subseteq \text{Assh}(M/M_1) \cup \{\mathfrak{m}\} = \text{Ass}(M/D_1) \cup \{\mathfrak{m}\}.$$

Thus $D_1/M_1 = U_{M/M_1}(0) = H_{\mathfrak{m}}^0(M/M_1)$. Hence D_1/M_1 has a finite length. Assume the result holds for i ; we will prove it for $i+1$. Since D_i/M_i has a finite length, we have

$$\text{Ass}(D_i/M_{i+1}) \subseteq \text{Ass}(M_i/M_{i+1}) \cup \{\mathfrak{m}\} = \text{Assh}(M_i/M_{i+1}) \cup \{\mathfrak{m}\}.$$

Thus $\text{Assh}(D_i/M_{i+1}) = \text{Assh}(M_i/M_{i+1})$ and so that $\text{Ass}(D_i/M_{i+1}) \subset \text{Assh}(D_i/M_{i+1}) \cup \{\mathfrak{m}\}$. Therefore, as similar in the case $i = 1$, D_{i+1}/M_{i+1} has a finite length. Hence D_i/M_i has a finite length for all $i = 1, \dots, t$. The claim $s = t$ follows from the definition of the dimension filtration and the fact that $\dim M_t \leq 0$. \square

Definition 2.3. (see [Sc]) Let $\mathcal{F} : M = M_0 \supset M_1 \supset \dots \supset M_t$ be a filtration of submodules of M . A system of parameters $\underline{x} = x_1, \dots, x_d$ of M is called a *distinguished system of parameters* of M with respect to \mathcal{F} , if $(x_{\dim M_{i+1}}, \dots, x_d) \subseteq \text{Ann } M_i$ for all positive integers i . A parameter ideal \mathfrak{q} of M is called a *distinguished parameter ideal* of M with respect to \mathcal{F} , if there exists a distinguished system of parameters $\underline{x} = x_1, \dots, x_d$ of M with respect to \mathcal{F} such that $\mathfrak{q} = (\underline{x})$. We simply say that $\mathfrak{q} = (\underline{x})$ is a distinguished parameter ideal if \underline{x} is a distinguished system of parameters with respect to the dimension filtration.

Let $\mathcal{F} : M = M_0 \supset M_1 \supset \dots \supset M_t$ be a filtration of submodules of M . For each submodule N of M , let $\mathcal{F}/N : M/N = (M_0+N)/N \supset (M_1+N)/N \supset \dots \supset (M_t+N)/N$ denote the filtration of submodules of M/N . When $N = xM$ for some $x \in R$, we abbreviate \mathcal{F}/xM to \mathcal{F}_x .

Lemma 2.4. Let $\mathcal{F} : M = M_0 \supset M_1 \supset \dots \supset M_t$ be a filtration of submodules of M . Then the following statements hold true.

- (1) A good system of parameters of M is also a distinguished system of parameters of M with respect to \mathcal{F} . Thus, there always exists a distinguished system of parameters with respect to \mathcal{F} .
- (2) Let N be a submodule of M . If x_1, \dots, x_d is a distinguished system of parameters of M with respect to \mathcal{F} and $\dim N < \dim M$, then x_1, \dots, x_d is a distinguished system of parameters of M/N with respect to \mathcal{F}/N .

Proof. Straightforward. \square

The following result of Y. Nakamura and the second author [GN] is often used in this section.

Lemma 2.5. [GN] Let R be a homomorphic image of a Cohen-Macaulay local ring and assume that $\text{Ass}(R) \subseteq \text{Assh}(R) \cup \{\mathfrak{m}\}$. Then

$$\mathcal{F} = \{\mathfrak{p} \in \text{Spec}(R) \mid \text{ht}_R(\mathfrak{p}) > 1 = \text{depth}(R_{\mathfrak{p}})\}$$

is a finite set.

The next proposition shows the existence of a special superficial element which is useful for many inductive proofs in the sequel.

Proposition 2.6. *Assume that R is a homomorphic image of a Cohen-Macaulay local ring. Let \mathfrak{q} be a parameter ideal of M . Then there exists an element $x \in \mathfrak{q}$ which is a superficial element of D_i with respect to \mathfrak{q} such that $\text{Ass}(\mathcal{D}_i/x\mathcal{D}_i) \subseteq \text{Assh}(\mathcal{D}_i/x\mathcal{D}_i) \cup \{\mathfrak{m}\}$, where $\mathcal{D}_i = D_i/D_{i+1}$ for all $i = 0, \dots, s-1$. Moreover, x is also a regular element of M/D_i for all $i = 1, \dots, s$.*

Proof. Set $I_i = \text{Ann}(\mathcal{D}_i)$, and $R_i = R/I_i$, then $\text{Ass}(R_i) = \text{Assh}(R_i)$ and $\dim R/I_i > \dim R/I_{i+1}$ for all $i = 0, \dots, s-1$. Moreover, we have

$$\text{Ass}(R_i) = \text{Ass}(\mathcal{D}_i) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \in \text{Ass}(M) \text{ and } \dim R/\mathfrak{p} = \dim R/I_i = d_i\}.$$

Set

$$\mathcal{F}_i = \{\mathfrak{p} \in \text{Spec}(R) \mid I_i \subset \mathfrak{p} \text{ and } \text{ht}_{R_i}(\mathfrak{p}/I_i) > 1 = \text{depth}((\mathcal{D}_i)_{\mathfrak{p}})\}.$$

By Lemma 2.5 and the fact $\text{Ass}(\mathcal{D}_i) = \text{Assh}(\mathcal{D}_i)$, we see that the set

$$\{\mathfrak{p} \in \text{Spec}(R_i) \mid \text{ht}_{R_i}(\mathfrak{p}) > 1 = \text{depth}((\mathcal{D}_i)_{\mathfrak{p}})\}$$

is finite, and so that \mathcal{F}_i are a finite set for all $i = 0, \dots, s-1$. Put $\mathcal{F} = \text{Ass}(M) \cup \bigcup_{i=1}^t \mathcal{F}_i \setminus \{\mathfrak{m}\}$. By the Prime Avoidance Theorem, we can choose $x \in \mathfrak{q} - \mathfrak{m}\mathfrak{q}$ such that x is a superficial element of D_i with respect to \mathfrak{q} such that $x \notin \bigcup_{\mathfrak{p} \in \mathcal{F}} \mathfrak{p}$. Since x is a superficial element of D_i and $\dim D_i > 0$ for all $i = 0, \dots, s-1$, $\dim \mathcal{D}_i/x\mathcal{D}_i = \dim \mathcal{D}_i - 1$. Let $\mathfrak{p} \in \text{Ass}(\mathcal{D}_i/x\mathcal{D}_i)$ with $\mathfrak{p} \neq \mathfrak{m}$. Then we have $\text{depth}(\mathcal{D}_i/x\mathcal{D}_i)_{\mathfrak{p}} = 0$. On the other hand, $\text{depth}(\mathcal{D}_i)_{\mathfrak{p}} > 0$ since $\mathfrak{p} \notin \text{Ass}(\mathcal{D}_i) \subseteq \text{Ass}(M)$. Hence $\text{depth}(\mathcal{D}_i)_{\mathfrak{p}} = 1$. It implies that $\text{ht}_{R_i}(\mathfrak{p}) = 1$, since $\mathfrak{p} \notin \mathcal{F}_i$. By the assumption R_i is a catenary ring, therefore

$$\dim R/\mathfrak{p} = \dim R_i - \text{ht}_{R_i}(\mathfrak{p}) = \dim R_i/xR_i = \dim \mathcal{D}_i/x\mathcal{D}_i.$$

Hence $\mathfrak{p} \in \text{Assh}(\mathcal{D}_i/x\mathcal{D}_i)$, and this completes the proof. \square

Lemma 2.7. *Let R , M and x be as in the Proposition 2.6 and $\mathcal{D}_{M/xM} : M/xM = D'_0 \supset D'_1 \supset \dots \supset D'_l$ the dimension filtration of M/xM . Then we have*

$$l = \begin{cases} s-1 & \text{if } \dim D_{s-1} = 1, \\ s & \text{otherwise.} \end{cases}$$

Moreover, D'_i/\overline{D}_i has a finite length, where $\overline{D}_i = (D_i + xM)/xM$, for all $i = 0, \dots, s-1$.

Proof. For a submodule N of M we set $\overline{N} = (N + xM)/xM$ a submodule of M/xM . For all $i = 1, \dots, s$, since x is a regular element of M/D_i , we have $D_i \cap xM = xD_i$, and so that

$$\begin{aligned} \mathcal{D}_i/x\mathcal{D}_i &\cong D_i/(xD_i + D_{i+1}) \cong D_i/[D_i \cap (xM + D_{i+1})] \\ &\cong (D_i + xM)/(D_{i+1} + xM) = \overline{D}_i/\overline{D}_{i+1}. \end{aligned}$$

Therefore, the filtration of submodules of M/xM

$$\mathcal{D}_x : M/xM = (D_0 + xM)/xM \supset (D_1 + xM)/xM \supset \dots \supset (D_s + xM)/xM$$

satisfies the following conditions: for all $i = 0, \dots, s-1$ and $\dim D_{i+1} > 0$, we have

$$\dim(D_i + xM)/xM > \dim(D_{i+1} + xM)/xM$$

and

$$\text{Ass}(M/xM) \setminus \{\mathfrak{m}\} \subseteq \bigcup_{i=0}^{s-1} \text{Ass}(\overline{D}_i/\overline{D}_{i+1}) \setminus \{\mathfrak{m}\}.$$

Thus, for all $\mathfrak{p} \in \text{Ass}(M/xM) \setminus \{\mathfrak{m}\}$, there is an integer i such that $\dim R/\mathfrak{p} = \dim(\overline{D}_i/\overline{D}_{i+1})$. Since $\mathcal{D}_{M/xM} : M/xM = D'_0 \supset D'_1 \supset \dots \supset D'_l$ is the dimension filtration of M/xM , it follows that either $l = s-1$ if $\dim D_{s-1} = 1$, or $l = s$ otherwise. Moreover, we also obtain $\dim \overline{D}_i = \dim D'_i$ and $\overline{D}_i \subseteq D'_i$ for all $i = 0, \dots, s-1$. Now we proceed by induction on i to show that D'_i/\overline{D}_i has a finite length for each $i = 1, \dots, s-1$. In fact, since $(D_1 + xM)/xM = \overline{D}_1 \subseteq D'_1$ and $\dim D'_1 < \dim M/xM$, $D'_1/\overline{D}_1 \subseteq U_{M/(D_1+xM)}(0)$. Moreover, by Lemma 2.6, we obtain $\text{Ass}(M/(D_1 + xM)) \subseteq \text{Assh}(M/(D_1 + xM)) \cup \{\mathfrak{m}\}$, and so that $U_{M/(D_1+xM)}(0)$ has a finite length. Hence D'_1/\overline{D}_1 has a finite length. Assume that the assertion holds for i , we will prove it for $i+1$. Since

$$\text{Ass}(\overline{D}_i/\overline{D}_{i+1}) = \text{Ass}((D_i + xM)/(D_{i+1} + xM)) = \text{Ass}(\mathcal{D}_i/x\mathcal{D}_i) \subseteq \text{Assh}(\mathcal{D}_i/x\mathcal{D}_i) \cup \{\mathfrak{m}\}$$

and $\text{Ass}(D'_i/\overline{D}_i) \subseteq \{\mathfrak{m}\}$ by the inductive hypothesis, we get

$$\text{Ass}(D'_i/\overline{D}_{i+1}) \subseteq \text{Ass}(\overline{D}_i/\overline{D}_{i+1}) \cup \text{Ass}(D'_i/\overline{D}_i) \subseteq \text{Assh}(\mathcal{D}_i/x\mathcal{D}_i) \cup \{\mathfrak{m}\}.$$

Therefore, it follows from the equality $\dim \mathcal{D}_i/x\mathcal{D}_i = \dim \overline{D}_i = \dim D'_i = \dim D'_i/\overline{D}_{i+1}$ that $\text{Ass}(D'_i/\overline{D}_{i+1}) \subseteq \text{Assh}(\overline{D}_i/\overline{D}_{i+1}) \cup \{\mathfrak{m}\} = \text{Assh}(D'_i/\overline{D}_{i+1}) \cup \{\mathfrak{m}\}$. Thus $U_{D'_i/\overline{D}_{i+1}}(0)$ has a finite length. Since $\overline{D}_{i+1} \subseteq D'_{i+1}$ and $\dim(D'_{i+1}/\overline{D}_{i+1}) < \dim D'_i/\overline{D}_{i+1}$, we have $D'_{i+1}/\overline{D}_{i+1} \subseteq U_{D'_i/\overline{D}_{i+1}}(0)$, and therefore $D'_{i+1}/\overline{D}_{i+1}$ has a finite length as required. \square

Corollary 2.8. *Let R , M , and x as in the Proposition 2.6. Then*

$$\Lambda(M/xM) = \{d_i - 1 \mid d_i = \dim D_i > 1, i = 0, \dots, s-1\}.$$

Corollary 2.9. *Let R , M and x be as in the Proposition 2.6. Let $\mathcal{F} : M = M_0 \supset M_1 \supset \dots \supset M_s$ be a filtration of submodules of M such that D_i/M_i has a finite length for each $i = 0, \dots, s$. For a submodule N of M we set $\overline{N} = N/xN$. Let $\mathcal{D}_{\overline{M}} : \overline{M} = D'_0 \supset D'_1 \supset \dots \supset D'_l$ be the dimension filtration of \overline{M} . Assume that there exists an integer $t_1 \leq s$ such that $\dim \overline{M}_{t_1} \leq 0$. Then the following conditions hold true.*

- (1) $t_1 = l$ and $\dim \overline{M}_i < \dim \overline{M}_{i-1}$ for all $i = 1, \dots, t_1$.
- (2) Either $t_1 = s-1$ if $\dim D_{s-1} = 1$, or $t_1 = s$ otherwise.
- (3) For each $i = 1, \dots, s-1$, D'_i/\overline{M}_i has a finite length.

Proof. (1) and (2) are trivial by Lemma 2.7.

(3). For each $i = 1, \dots, s-1$, since M_i is submodule of D_i and D_i/M_i has a finite length, $\overline{M}_i \subset \overline{D}_i$ and $\overline{D}_i/\overline{M}_i$ has a finite length. By Lemma 2.7, D'_i/\overline{D}_i has a finite length and so has D'_i/\overline{M}_i . \square

Lemma 2.10. *Let R , M , \mathfrak{q} and x be as in the Proposition 2.6. Let $\mathcal{F} : M = M_0 \supset M_1 \supset \dots \supset M_s$ be a filtration of submodules of M such that D_i/M_i has a finite length for each $i = 0, \dots, s$. Assume that \mathfrak{q} is a distinguished parameter ideal of M with respect to \mathcal{F} . Then there exists a distinguished system of parameters x_1, \dots, x_d of M with respect to \mathcal{F} such that $x_1 = x$ and $\mathfrak{q} = (x_1, \dots, x_d)$.*

Proof. Since \mathfrak{q} is a distinguished parameter ideal of M with respect to \mathcal{F} , there exists a distinguished system of parameters y_1, \dots, y_d of M with respect to \mathcal{F} such that $\mathfrak{q} = (y_1, \dots, y_d)$ and $M_i \subseteq 0 :_M y_j$ for all $j = d_i + 1, \dots, d$ and $i = 1, \dots, s$. In particular, we have $(y_{d_{s-1}+1}, \dots, y_d) \subseteq \text{Ann } M_{s-1}$. Moreover, by Lemma 2.2 we have $\dim D_{s-1} > 0$ and $\text{Assh } M_{s-1} = \text{Assh } D_{s-1} = \{\mathfrak{p} \in \text{Ass}(M) \mid \dim R/\mathfrak{p} = d_{s-1}\}$. Thus $(y_{d_{s-1}+1}, \dots, y_d) \subseteq \bigcap_{\mathfrak{p} \in \text{Ass}(M), \dim R/\mathfrak{p} = d_{s-1}} \mathfrak{p}$. Since $\dim D_{s-1} > 0$ and by the choice of x , the elements $x, y_{d_{s-1}+1}, \dots, y_d$ form a part of a minimal basis of \mathfrak{q} . Thus $x, y_{d_{s-1}+1}, \dots, y_d$ is a part of a system of parameters of M . Therefore we can find d_{s-1} elements $x_1 = x, x_2, \dots, x_{d_{s-1}}$ in \mathfrak{q} that such $\mathfrak{q} = (x_1, x_2, \dots, x_{d_{s-1}}, x_{d_{s-1}+1} = y_{d_{s-1}+1}, \dots, x_d = y_d)$ as required. \square

3. ARITHMETIC DEGREE AND HILBERT COEFFICIENTS

For prime ideal \mathfrak{p} of R , we define the length-multiplicity of M at \mathfrak{p} as the length of $R_{\mathfrak{p}}$ -module $\Gamma_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = H_{\mathfrak{p}R_{\mathfrak{p}}}^0(M_{\mathfrak{p}})$ and denote it by $\text{mult}_M(\mathfrak{p})$. It is easy to see that $\text{mult}_M(\mathfrak{p}) \neq 0$ if and only if \mathfrak{p} is an associated prime of M .

Definition 3.1. ([BM],[V],[V1]) Let I be an \mathfrak{m} -primary ideal and i a non-negative integer. We define the i -th arithmetic degree of M with respect to I by

$$\text{arith-deg}_i(I, M) = \sum_{\mathfrak{p} \in \text{Ass}(M), \dim R/\mathfrak{p} = i} \text{mult}_M(\mathfrak{p}) e_0(I, R/\mathfrak{p}).$$

The arithmetic degree of M with respect to I is the integer

$$\begin{aligned} \text{arith-deg}(I, M) &= \sum_{\mathfrak{p} \in \text{Ass}(M)} \text{mult}_M(\mathfrak{p}) e_0(I, R/\mathfrak{p}) \\ &= \sum_{i=0}^d \text{arith-deg}_i(I, M). \end{aligned}$$

The following result gives a relationship between the multiplicity of submodules in the dimension filtration and the arithmetic degree.

Proposition 3.2. *Let (R, \mathfrak{m}) be a local Noetherian ring, I an \mathfrak{m} -primary ideal and $\mathcal{D} : M = D_0 \supset D_1 \supset \dots \supset D_s = H_{\mathfrak{m}}^0(M)$ the dimension filtration of R -module M . Then the following statements hold true.*

- (1) $\text{arith-deg}_0(I, M) = \ell_R(H_{\mathfrak{m}}^0(M))$.
- (2) For $j = 1, \dots, d$, we have

$$\text{arith-deg}_j(I, M) = \begin{cases} e_0(I, D_i) & \text{if } j = \dim D_i \in \Lambda(M), \text{ some } i \\ 0 & \text{if } j \notin \Lambda(M). \end{cases}$$

Proof. (1) is trivial from the definition of the arithmetic degree.

(2). By the associativity formula for multiplicities, we have

$$e_0(I, D_i) = \sum_{\mathfrak{p} \in \text{Ass } D_i, \dim R/\mathfrak{p} = d_i} \ell((D_i)_{\mathfrak{p}}) e_0(I, R/\mathfrak{p}).$$

It follows from $\{\mathfrak{p} \in \text{Ass}(D_i) \mid \dim R/\mathfrak{p} = d_i\} = \{\mathfrak{p} \in \text{Ass}(M) \mid \dim R/\mathfrak{p} = d_i\}$ that $H_{\mathfrak{p}R_{\mathfrak{p}}}^0(M_{\mathfrak{p}}) \cong (D_i)_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Ass}(M)$ with $\dim R/\mathfrak{p} = d_i$. Thus we get

$$\ell((D_i)_{\mathfrak{p}}) = \ell(H_{\mathfrak{p}R_{\mathfrak{p}}}^0(M_{\mathfrak{p}})) = \text{mult}_M(\mathfrak{p})$$

for all $\mathfrak{p} \in \text{Ass}(M)$ and $\dim R/\mathfrak{p} = d_i$. Hence

$$e_0(I, D_i) = \text{arith-deg}_{d_i}(I, M),$$

for all $i = 0, \dots, s$. The rest of the proposition is trivial. \square

For proving the main result in next section, we need two auxiliary lemmas as follows. It should be noticed that the statement (1) of Lemma 3.3 below is also shown in [MSV], but the proof here is shorter.

Lemma 3.3. *Let \mathfrak{q} be a parameter ideal of M with $\dim M = d$. Then the following statements hold true.*

- (1) If $d = 1$, then $e_1(\mathfrak{q}, M) = -\ell_R(H_{\mathfrak{m}}^0(M))$.
- (2) If $d \geq 2$, then for every superficial element $x \in \mathfrak{q}$ of M it holds

$$e_j(\mathfrak{q}, M) = \begin{cases} e_j(\mathfrak{q}, M/xM) & \text{if } 0 \leq j \leq d-2, \\ e_{d-1}(\mathfrak{q}, M/xM) + (-1)^{d-1} \ell_R(0 :_M x) & \text{if } j = d-1, \end{cases}$$

Proof. Let $d = 1$ and $\mathfrak{q} = (a)$. Choose the integer n large enough such that $H_{\mathfrak{m}}^0(M) = 0 :_M a^n$ and $\ell(M/a^n M) = e_0((a), M)n - e_1((a), M)$. Then

$$e_1((a), M) = -(\ell(M/a^n M) - e_0((a^n), M)) = -\ell(0 :_M a^n) = -\ell(H_{\mathfrak{m}}^0(M)).$$

The second statement was proved by M. Nagata [N, 22.6]. \square

Lemma 3.4. *Let N be a submodule of M with $\dim N = s < d$ and I an \mathfrak{m} -primary ideal of R . Then*

$$e_j(I, M) = \begin{cases} e_j(I, M/N) & \text{if } 0 \leq j \leq d-s-1, \\ e_{d-s}(I, M/N) + (-1)^{d-s} e_0(I, N) & \text{if } j = d-s. \end{cases}$$

Proof. From the exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$$

we get the following exact sequence

$$0 \rightarrow (N \cap I^n M)/I^n N \rightarrow N/I^n N \rightarrow M/I^n M \rightarrow M/I^n M + N \rightarrow 0$$

for each n . Thus

$$\ell(M/I^n M) = \ell(N/I^n N) + \ell(M/I^n M + N) - \ell((N \cap I^n M)/I^n N)$$

for all n . Hence $\ell((N \cap I^n M)/I^n N)$ is a polynomial for large enough n . By the Artin-Rees lemma, there exists an integer k such that $N \cap I^n M \subseteq I^{n-k} N$ for all $n \geq k$, and so that

$$\ell((N \cap I^n M)/I^n N) \leq \ell(I^{n-k} N/I^n N) \leq \sum_{i=n-k}^{n-1} \ell(I^i N/I^{i+1} N)$$

for all $n \geq k$. This gives that the degree of the polynomial $\ell((N \cap I^n M)/I^n N)$ is strictly smaller than $\dim N$. Since $\dim N = s < d$, the conclusion follows by comparing coefficients of polynomials in the above equality. \square

4. CHARACTERIZATION OF SEQUENTIALLY COHEN-MACAULAY MODULES

The notion of sequentially Cohen-Macaulay module was introduced first by Stanley [St] for graded case and in [Sc], [CN] for the local case.

Definition 4.1. An R -module M is called a *sequentially Cohen-Macaulay module* if there exists a filtration $\mathcal{F} : M = M_0 \supset M_1 \supset \dots \supset M_t$ of submodules of M such that $\dim M_t \leq 0$, $\dim M_{i+1} < \dim M_i$ and $\mathcal{M}_i = M_i/M_{i+1}$ are a Cohen-Macaulay module for all $i = 0, \dots, t-1$.

It should be noticed here that if M is a sequentially Cohen-Macaulay, the filtration \mathcal{F} in the definition above is uniquely determined and it is just the dimension filtration $\mathcal{D} : M = D_0 \supset D_1 \supset \dots \supset D_s = H_{\mathfrak{m}}^0(M)$ of M . Therefore, M is always a sequentially Cohen-Macaulay module, if $\dim M = 1$. Now we give a characterization of sequentially Cohen-Macaulay modules having small dimension.

Theorem 4.2. *Let M be a finitely generated R -module with $\dim M = 2$. Then the following statements are equivalent:*

- (1) M is a sequentially Cohen-Macaulay R -module.
- (2) For all parameter ideals \mathfrak{q} of M and $j = 0, 1, 2$, we have

$$e_j(\mathfrak{q}, M) = (-1)^j \text{arith-deg}_{2-j}(\mathfrak{q}, M).$$

- (3) For all parameter ideals \mathfrak{q} of M , we have

$$e_1(\mathfrak{q}, M) = -\text{arith-deg}_1(\mathfrak{q}, M).$$

(4) For some parameter ideal \mathfrak{q} of M , we have

$$e_1(\mathfrak{q}, M) = -\text{arith-deg}_1(\mathfrak{q}, M).$$

Proof. (1) \Rightarrow (2). The result follows from the Propositions 3.2 and 3.4.

(2) \Rightarrow (3) and (3) \Rightarrow (4) are trivial.

(4) \Rightarrow (1). It suffices to show that $\overline{M} = M/D_1$ is a Cohen-Macaulay module. In fact, since $\dim D_1 < \dim M = 2$, then $\dim D_1 = 0$ or 1. If $\dim D_1 = 0$, then $\text{arith-deg}_1(\mathfrak{q}, M) = 0$. Therefore we get by Lemma 3.4 and the hypothesis that

$$e_1(\mathfrak{q}, \overline{M}) = e_1(\mathfrak{q}, M) = 0.$$

If $\dim D_1 = 1$, it follows from Lemma 3.4 and Proposition 3.2 that

$$e_1(\mathfrak{q}, \overline{M}) = e_1(\mathfrak{q}, M) + e_0(\mathfrak{q}, D_1) = e_1(\mathfrak{q}, M) + \text{arith-deg}_1(\mathfrak{q}, M) = 0.$$

Thus in all cases we have $e_1(\mathfrak{q}, \overline{M}) = 0$. Choose now an element $x \in \mathfrak{q}$ which is a superficial element of \overline{M} with respect to \mathfrak{q} . Then x is an \overline{M} -regular element, since $\text{Ass } \overline{M} = \text{Assh } \overline{M}$. It follows from the assumption $\dim \overline{M} = 2$ and Lemma 3.3 that

$$0 = e_1(\mathfrak{q}, \overline{M}) = e_1(\mathfrak{q}, \overline{M}/x\overline{M}) = -\ell(H_{\mathfrak{m}}^0(\overline{M}/x\overline{M})).$$

Thus $H_{\mathfrak{m}}^0(\overline{M}/x\overline{M}) = 0$. So $\text{depth } \overline{M} = 2$ and \overline{M} is a Cohen-Macaulay module. \square

Proposition 4.3. Let $\mathcal{F} : M = M_0 \supset M_1 \supset \dots \supset M_s$ be a filtration of submodules of M such that D_i/M_i has a finite length for each $i = 0, \dots, s$, where s is the length of the dimension filtration of M . Assume that M/D_j is a sequentially Cohen-Macaulay module for some $1 \leq j \leq s$ and x_1, \dots, x_d is a distinguished system of parameters of M with respect to \mathcal{F} . Set $\mathfrak{q} = (x_1, \dots, x_d)$. Then the following statements hold true.

(1) For all $i = 1, \dots, j$, we have

$$(x_1, \dots, x_d)^{n+1}M \cap D_i = (x_1, \dots, x_d)^{n+1}D_i,$$

for large enough n .

(2) We have

$$\ell(M/\mathfrak{q}^{n+1}M) = \sum_{i=0}^{j-1} \binom{n+d_i}{d_i} e_0(\mathfrak{q}, D_i) + \ell(D_j/\mathfrak{q}^{n+1}D_j),$$

for large enough n .

Proof. (1). Let $j \leq s$ be a positive integer and M/D_j a sequentially Cohen-Macaulay module. We prove statement (1) recursively on $i \leq j$. Let $i = 1$. Since M/D_j is a sequentially Cohen-Macaulay module, M/D_1 is Cohen-Macaulay. Thus $(x_1, \dots, x_d)^k M \cap D_1 = (x_1, \dots, x_d)^k D_1$ for all k . Since D_1/M_1 is of finite length, there exists a positive integer n such that $(x_1, \dots, x_d)^n D_1 \subseteq M_1$. On the other hand, since x_1, \dots, x_d is a

distinguished system of parameters of M with respect to \mathcal{F} , $(x_{d_1+1}, \dots, x_d)M_1 = 0$. It follows for large enough n that

$$\begin{aligned} (x_1, \dots, x_d)^{n+1}M \cap D_1 &= (x_1, \dots, x_d)^{n+1}D_1 \\ &= (x_1, \dots, x_{d_1})^{n+1}D_1 + (x_{d_1+1}, \dots, x_d)(x_1, \dots, x_d)^n D_1 \\ &\subseteq (x_1, \dots, x_{d_1})^{n+1}D_1 + (x_{d_1+1}, \dots, x_d)M_1 \\ &= (x_1, \dots, x_{d_1})^{n+1}D_1. \end{aligned}$$

Therefore we get $(x_1, \dots, x_d)^{n+1}M \cap D_1 = (x_1, \dots, x_{d_1})^{n+1}D_1$. Assume now that the conclusion is true for $i-1 < j$. Then we get

$$\begin{aligned} (x_1, \dots, x_d)^{n+1}M \cap D_i &= ((x_1, \dots, x_d)^{n+1}M \cap D_{i-1}) \cap D_i \\ &= (x_1, \dots, x_{d_{i-1}})^{n+1}D_{i-1} \cap D_i. \end{aligned}$$

Consider now the module D_{i-1} with two filtrations of submodules $\mathcal{F}' : D_{i-1} \supset M_i \supset \dots \supset M_s$ and the dimension filtration $\mathcal{D}' : D_{i-1} \supset D_i \supset \dots \supset D_s$. It is easy to check that the module D_{i-1} with these two filtrations of submodules satisfies all of assumptions of the proposition. Thus, by applying our proof for the case $i = 1$ with the notice that $x_1, \dots, x_{d_{i-1}}$ is a distinguished system of parameters of D_{i-1} with respect to \mathcal{F}' we have

$$(x_1, \dots, x_d)^{n+1}M \cap D_i = (x_1, \dots, x_{d_{i-1}})^{n+1}D_{i-1} \cap D_i = (x_1, \dots, x_{d_i})^{n+1}D_i,$$

for large enough n , which finishes the proof of statement (1).

(2) We argue by the induction on the length s of the dimension filtration \mathcal{D} of M . The case $s = 0$ is obvious. Assume that $s \geq j > 0$. By virtue of the statement (1) we get a short exact sequence

$$0 \rightarrow D_1/\mathfrak{q}^{n+1}D_1 \rightarrow M/\mathfrak{q}^{n+1}M \rightarrow M/\mathfrak{q}^{n+1}M + D_1 \rightarrow 0$$

for large enough n . Therefore we have

$$\ell(M/\mathfrak{q}^{n+1}M) = \ell(D_1/(x_1, \dots, x_{d_1})^{n+1}D_1) + \ell(\mathcal{D}_0/\mathfrak{q}^{n+1}\mathcal{D}_0),$$

where $\mathcal{D}_0 = M/D_1$. Since x_1, \dots, x_d is a distinguished system of parameters of M with respect to \mathcal{F} , x_1, \dots, x_{d_1} is a distinguished system of parameters of D_1 with respect to the filtration $D_1 \supset M_2 \supset \dots \supset M_s$. Notice that $D_1 \supset D_2 \supset \dots \supset D_s$ is the dimension filtration of D_1 and D_k/M_k has a finite length for each $k = 1, \dots, s$. Since $s \geq j > 0$ and M/D_j is a sequentially Cohen-Macaulay module, so is D_1/D_j . Because the dimension filtration of D_1 is of the length $s - 1$, it follows from the inductive hypothesis that

$$\ell(D_1/(x_1, \dots, x_{d_1})^{n+1}D_1) = \sum_{i=1}^{j-1} \binom{n+d_i}{d_i} e_0(\mathfrak{q}, D_i) + \ell(D_j/\mathfrak{q}^{n+1}D_j).$$

Since $\mathcal{D}_0 = M/D_1$ is Cohen-Macaulay of dimension $d = d_0$, we have

$$\ell(\mathcal{D}_0/\mathfrak{q}^{n+1}\mathcal{D}_0) = \binom{n+d}{d} e_0(\mathfrak{q}, \mathcal{D}_0) = \binom{n+d}{d} e_0(\mathfrak{q}, D_0).$$

Hence

$$\ell(M/\mathfrak{q}^{n+1}M) = \sum_{i=0}^{j-1} \binom{n+d_i}{d_i} e_0(\mathfrak{q}, D_i) + \ell(D_j/\mathfrak{q}^{n+1}D_j),$$

for all large enough $n \geq 0$ as required. \square

Proposition 4.4. *Let R be a homomorphic image of a Cohen-Macaulay local ring and M a finitely generated R module of dimension $d = \dim M \geq 2$. Let $\mathcal{F} : M = M_0 \supset M_1 \supset \dots \supset M_s$ be a filtration of submodules of M such that D_i/M_i has a finite length for each $i = 1, \dots, s$. Assume that \mathfrak{q} is a distinguished parameter ideal of M with respect to \mathcal{F} such that for all $j \in \Lambda(M)$ we have*

$$e_{d-j+1}(\mathfrak{q}, M) = (-1)^{d-j+1} \text{arith-deg}_{j-1}(\mathfrak{q}, M).$$

Then M is a sequentially Cohen-Macaulay module.

Proof. Reminder that $\mathcal{D} : M = D_0 \supset D_1 \supset \dots \supset D_s = H_{\mathfrak{m}}^0(M)$ is the dimension filtration of M and $\Lambda(M) = \{d_i = \dim D_i \mid i = 1, \dots, s-1\}$. For each $i = 0, \dots, s-1$, we set

$$\hat{e}_0(\mathfrak{q}, D_{i+1}) = \begin{cases} e_0(\mathfrak{q}, D_{i+1}) & \text{if } d_{i+1} = d_i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by virtue of Proposition 3.2 the equality in the assumptions of our proposition can be rewritten as

$$e_{d-d_i+1}(\mathfrak{q}, M) = (-1)^{d-d_i+1} \hat{e}_0(\mathfrak{q}, D_{i+1}) \quad (*)$$

for all $i = 0, \dots, s-1$. We prove a statement which is slight stronger than the proposition, but it is more convenient for the inductive process as follows: M is sequentially Cohen-Macaulay if the equations $(*)$ hold true for all $d_i \in \Lambda(M)$ with $d_i > 1$. We proceed by induction on d . The claim is proved for the case $d = 2$ by Theorem 4.2.

Suppose that $d \geq 3$. Then there exists by Proposition 2.6 an element $x \in \mathfrak{q}$ which is a superficial element of D_i with respect to \mathfrak{q} such that x is a regular element of M/D_i for all $i = 1, \dots, s$. For a submodule N of M , we denote $\overline{N} = (N + xM)/xM$ the submodule of M/xM . Let $\mathcal{D}_{M/xM} : M/xM = D'_0 \supset D'_1 \supset \dots \supset D'_l$ be the dimension filtration of M/xM and $t \in \{0, \dots, s\}$ an integer such that $\dim \overline{M}_t \leq 0$. By Corollaries 2.8, 2.9 and Lemma 2.10, the filtration $\mathcal{F}_x : M/xM = \overline{M}_0 \supset \overline{M}_1 \supset \dots \supset \overline{M}_t$ of submodules of M/xM satisfies the following conditions:

- (1) Either $t = l = s-1$ if $\dim D_{s-1} = 1$, or $t = l = s$ otherwise.
- (2) For each $i = 1, \dots, s-1$, D'_i/\overline{M}_i has a finite length.
- (3) For each $i = 1, \dots, s-1$, D'_i/\overline{D}_i has a finite length and

$$\Lambda(M/xM) = \{d_i - 1 \mid d_i > 1, i = 1, \dots, s-1\}.$$

- (4) There exists a distinguished system of parameters x_1, \dots, x_d of M with respect to \mathcal{F} such that $x_1 = x$ and $\mathfrak{q} = (x_1, \dots, x_d)$. Moreover, the system of parameters x_2, \dots, x_d of M/xM is a distinguished system of parameters of M/xM with respect to \mathcal{F}_x .

Now, we first show that the module $\overline{M} = M/xM$ satisfies all the assumptions of the proposition with the filtrations of submodules $\mathcal{D}_{M/xM}$, \mathcal{F}_x and the distinguished parameter ideal (x_2, \dots, x_d) with respect to \mathcal{F}_x . Since x is a regular element of M/D_i for all $i = 1, \dots, s$, we have $D_i \cap xM = xD_i$. Therefore

$$\mathcal{D}_i/x\mathcal{D}_i \cong D_i/xD_i + D_{i+1} \cong D_i/[D_i \cap (xM + D_{i+1})] \cong (D_i + xM)/(D_{i+1} + xM).$$

It follows that if $d_i > 1$ then $e_0(\mathfrak{q}, \mathcal{D}_i/x\mathcal{D}_i) = e_0(\mathfrak{q}, \overline{D}_i/\overline{D}_{i+1}) = e_0(\mathfrak{q}, \overline{D}_i) = e_0(\mathfrak{q}, D'_i)$ as D'_i/\overline{D}_i is of finite length, and so that

$$e_0(\mathfrak{q}, D_i) = e_0(\mathfrak{q}, \mathcal{D}_i/x\mathcal{D}_i) = e_0(\mathfrak{q}, D'_i),$$

since x is also a non-zero divisor on \mathcal{D}_i . Let $d_i - 1 \in \Lambda(M/xM)$, and $d_i > 2$. We consider the following two cases: If $d_{i+1} = d_i - 1 > 1$, and $\dim D'_{i+1} = \dim D_{i+1} - 1 = \dim D_i - 2 = \dim D'_i - 1$, therefore $\hat{e}_0(\mathfrak{q}, D'_{i+1}) = e_0(\mathfrak{q}, D'_{i+1})$. Then, by applying Lemma 3.3, Proposition 3.2 we get that

$$\begin{aligned} e_{(d-1)-(d_i-1)+1}(\mathfrak{q}, M/xM) &= e_{d-d_{i+1}}(\mathfrak{q}, M) \\ &= (-1)^{d-d_{i+1}} e_0(\mathfrak{q}, D_{i+1}) \\ &= (-1)^{d-d_{i+1}} e_0(\mathfrak{q}, D'_{i+1}) \\ &= (-1)^{(d-1)-(d_i-1)+1} \hat{e}_0(\mathfrak{q}, D'_{i+1}). \end{aligned}$$

If $d_{i+1} \neq d_i - 1$, $\dim D'_{i+1} \neq \dim D'_i - 1$, and so that

$$\hat{e}_0(\mathfrak{q}, D_{i+1}) = \hat{e}_0(\mathfrak{q}, D'_{i+1}) = 0.$$

Thus

$$e_{(d-1)-(d_i-1)+1}(\mathfrak{q}, M/xM) = (-1)^{(d-1)-(d_i-1)+1} \hat{e}_0(\mathfrak{q}, D'_{i+1}) = 0.$$

This show that in both cases we obtain

$$e_{(d-1)-(d_i-1)+1}(\mathfrak{q}, M/xM) = (-1)^{(d-1)-(d_i-1)+1} \hat{e}_0(\mathfrak{q}, D'_{i+1})$$

for all $d_i - 1 \in \Lambda(M/xM)$ and $d_i - 1 > 1$. Therefore M/xM is a sequentially Cohen-Macaulay module by the inductive hypothesis.

Next, we prove by induction on i that for all $i = 0, \dots, s-1$, if $d_i \geq 3$ then $D'_{i+1} = \overline{D}_{i+1}$ and D_i/D_{i+1} is a Cohen-Macaulay module. In fact, let $i = 0$. Since \overline{M}/D'_1 is a Cohen-Macaulay module and D'_1/\overline{D}_1 has a finite length, $H_{\mathfrak{m}}^i(M/D_1 + xM) = 0$ for all $0 < i < d - 1$. Therefore, we derive from exact sequence

$$0 \rightarrow M/D_1 \xrightarrow{x} M/D_1 \rightarrow M/D_1 + xM \rightarrow 0$$

the following exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^0(M/D_1 + xM) \rightarrow H_{\mathfrak{m}}^1(M/D_1) \xrightarrow{x} H_{\mathfrak{m}}^1(M/D_1) \rightarrow 0.$$

Thus $H_{\mathfrak{m}}^1(M/D_1) = 0$, and so $D'_1/\overline{D}_1 = H_{\mathfrak{m}}^0(M/D_1 + xM) = 0$. Hence $D'_1 = \overline{D}_1$. Moreover, since x is $\mathcal{D}_0 = M/D_1$ -regular and $\mathcal{D}_0/x\mathcal{D}_0 \cong \overline{M}/\overline{D}_1 = \overline{M}/D'_1$ a Cohen-Macaulay module, \mathcal{D}_0 is a Cohen-Macaulay module. Assume now that $D'_j = \overline{D}_j$ and D_j/D_{j+1} are Cohen-Macaulay for all $j \leq i$ with $d_i \geq 3$. Then with the same argument as above, we can prove that $H_{\mathfrak{m}}^j(\mathcal{D}_i/x\mathcal{D}_i) = 0$ for all $0 < j < d_i - 1$. Therefore, from the exact sequence

$$0 \rightarrow \mathcal{D}_i \xrightarrow{x} \mathcal{D}_i \rightarrow \mathcal{D}_i/x\mathcal{D}_i \rightarrow 0$$

we obtain the following exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^0(\mathcal{D}_i/x\mathcal{D}_i) \rightarrow H_{\mathfrak{m}}^1(\mathcal{D}_i) \xrightarrow{x} H_{\mathfrak{m}}^1(\mathcal{D}_i) \rightarrow 0.$$

It follows that $H_{\mathfrak{m}}^1(\mathcal{D}_i) = 0$. Therefore $D'_{i+1} = \overline{D}_{i+1}$. The Cohen-Macaulayness of D_i/D_{i+1} follows from the fact that $\mathcal{D}_i/x\mathcal{D}_i \cong \overline{D}_i/\overline{D}_{i+1} = D'_i/D'_{i+1}$ is a Cohen-Macaulay module.

Denote by N the largest submodule of M such that $\dim N \leq 2$. It should be mentioned that this submodule N must be appeared in the dimension filtration of M , says $N = D_k$ for some $k \in \{s-2, s-1, s\}$. Then, from the proof above it is easy to see that if $\dim N \leq 1$, M is sequentially Cohen-Macaulay. Assume that $\dim N = 2$. To prove M is sequentially Cohen-Macaulay in this case, it remains to show that $N = D_k$ is sequentially Cohen-Macaulay. By virtue of Lemma 4.3 we have for large enough n

$$\ell(M/\mathfrak{q}^{n+1}M) = \sum_{i=0}^{k-1} \binom{n+d_i}{d_i} e_0(\mathfrak{q}, D_i) + \ell(N/\mathfrak{q}^{n+1}N).$$

Therefore by comparing coefficients of the equality above and by hypotheses of the proposition we get

$$\begin{aligned} -e_1(\mathfrak{q}, N) &= (-1)^{d-1} e_{d-1}(\mathfrak{q}, M) \\ &= (-1)^{d-1} e_{d-2+1}(\mathfrak{q}, M) \\ &= (-1)^{d-1} (-1)^{d-2+1} \text{arith-deg}_{2-1}(\mathfrak{q}, M) \\ &= \text{arith-deg}_1(\mathfrak{q}, M) = \text{arith-deg}_1(\mathfrak{q}, N). \end{aligned}$$

Thus N is a sequentially Cohen-Macaulay module by Theorem 4.2, and the proof of the proposition is complete. \square

We are now able to state our main result.

Theorem 4.5. *Assume that R is a homomorphic image of a Cohen-Macaulay local ring. Then the following statements are equivalent:*

- (1) M is a sequentially Cohen-Macaulay R -module.

(2) For all distinguished parameter ideals \mathfrak{q} of M and $j = 0, \dots, d$, we have

$$e_j(\mathfrak{q}, M) = (-1)^j \text{arith-deg}_{d-j}(\mathfrak{q}, M).$$

(3) For all distinguished parameter ideals \mathfrak{q} of M and $j \in \Lambda(M)$, we have

$$e_{d-j+1}(\mathfrak{q}, M) = (-1)^{d-j+1} \text{arith-deg}_{j-1}(\mathfrak{q}, M).$$

(4) For some distinguished parameter ideal \mathfrak{q} of M and for all $j \in \Lambda(M)$, we have

$$e_{d-j+1}(\mathfrak{q}, M) = (-1)^{d-j+1} \text{arith-deg}_{j-1}(\mathfrak{q}, M).$$

Proof. (1) \Rightarrow (2). Since M is a sequentially Cohen-Macaulay module, it follows from Proposition 4.3 with $j = s$ that

$$\ell(M/\mathfrak{q}^{n+1}M) = \sum_{i=0}^s \binom{n+d_i}{d_i} e_0(\mathfrak{q}, D_i)$$

for all distinguished parameter ideals \mathfrak{q} and large enough n . Therefore we get

$$(-1)^{d-d_i} e_{d-d_i}(\mathfrak{q}, M) = e_0(\mathfrak{q}, D_i)$$

for all $i = 0, \dots, s$ and $e_j(\mathfrak{q}, M) = 0$ for all $j \neq d - d_i$. Therefore the conclusion follows from the Proposition 3.2.

(2) \Rightarrow (3) and (3) \Rightarrow (4) are trivial.

(4) \Rightarrow (1) follows from the Proposition 4.4. \square

The first consequence of Theorem 4.5 is to give an affirmative answer for Vasconcelos' Conjecture announced in the introduction. It is noticed that recently this conjecture has been settled in [GGHOPV] and extended for modules in [MSV, 3.11] provided $\dim R = \dim M$.

Corollary 4.6. *Suppose that M is an unmixed R -module, that is $\dim(\hat{R}/P) = \dim M$ for all $P \in \text{Ass}_{\hat{R}} \hat{M}$, where \hat{M} is the \mathfrak{m} -adic completion of M . The M is a Cohen-Macaulay module if and only if $e_1(\mathfrak{q}, M) \geq 0$ for some parameter ideal \mathfrak{q} of M .*

Proof. Since M is unmixed, we may assume without loss of generality that R is complete. Therefore R is a homomorphic image of a Cohen-Macaulay local ring and $M = D_0 \supset D_1 = 0$ is the dimension filtration of M . Thus $\Lambda(M) = \{d\}$ and M is Cohen-Macaulay if it is sequentially Cohen-Macaulay. It follows from Theorem 4.5 and the fact that every parameter ideal of M is good that M is Cohen-Macaulay if and only if there exists a parameter ideal \mathfrak{q} such that $e_1(\mathfrak{q}, M) = -\text{arith-deg}_{d-1}(\mathfrak{q}, M)$. And the last condition is equivalent to the condition $e_1(\mathfrak{q}, M) \geq 0$ by Proposition 3.2. \square

The next corollary shows that the Chern number of a parameter ideal \mathfrak{q} is not only a non-positive integer but also bounded above by $-\text{arith-deg}_{d-1}(\mathfrak{q}, M)$.

Corollary 4.7. *Let M be a finitely generated R -module of dimension $d > 0$. Then $e_1(\mathfrak{q}, M) \leq -\text{arith-deg}_{d-1}(\mathfrak{q}, M)$ for all parameter ideals \mathfrak{q} of M .*

Proof. Since the Hilbert coefficients and arithmetic degrees are unchanged by the \mathfrak{m} -adic completion, we can assume that R is complete. Then by Lemma 3.4 we have

$$e_1(\mathfrak{q}, M) + \text{arith-deg}_{d-1}(\mathfrak{q}, M) = e_1(\mathfrak{q}, M/U_M(0)),$$

where $U_M(0) = D_{s-1}$ is the unmixed part of M . If $e_1(\mathfrak{q}, M) \geq -\text{arith-deg}_{d-1}(\mathfrak{q}, M)$, $e_1(\mathfrak{q}, M/U_M(0)) \geq 0$. Then $M/U_M(0)$ is Cohen-Macaulay by Corollary 4.6, and so that $e_1(\mathfrak{q}, M/U_M(0)) = 0$. Hence $e_1(\mathfrak{q}, M) \leq -\text{arith-deg}_{d-1}(\mathfrak{q}, M)$ for all parameter ideals \mathfrak{q} of M . \square

The following immediate consequence of 4.7 is first proved in [MSV, Theorem 3.5].

Corollary 4.8. *Let M be a finitely generated R -module of dimension $d > 0$. Then $e_1(\mathfrak{q}, M) \leq 0$ for all parameter ideals \mathfrak{q} of M .*

Below, we give some more corollaries in the cases that Chern numbers of parameter ideals have extremal values.

Corollary 4.9. *Assume that R is a homomorphic image of a Cohen-Macaulay local ring. Then the following assertions are equivalent:*

- (1) $e_1(\mathfrak{q}, M) = -\text{arith-deg}_{d-1}(\mathfrak{q}, M)$ for all parameter ideals \mathfrak{q} of M .
- (2) $e_1(\mathfrak{q}, M) = -\text{arith-deg}_{d-1}(\mathfrak{q}, M)$ for some parameter ideal \mathfrak{q} of M .
- (3) $M/U_M(0)$ is a Cohen-Macaulay module.

Proof. It follows immediately from Theorem 4.5 and the fact that if M is unmixed then every parameter ideal of M is good. \square

By virtue of Proposition 3.2 we see that $\text{arith-deg}_{d-1}(\mathfrak{q}, M) = 0$ for some parameter ideal \mathfrak{q} of M if and only if $\dim U_M(0) \leq d - 2$. Hence from this fact and Corollary 4.9 we have

Corollary 4.10. *Assume that R is a homomorphic image of a Cohen-Macaulay local ring. The following assertions are equivalent:*

- (1) $e_1(\mathfrak{q}, M) = 0$ for all parameter ideals \mathfrak{q} of M .
- (2) $e_1(\mathfrak{q}, M) = 0$ for some parameter ideal \mathfrak{q} of M .
- (3) $M/U_M(0)$ is Cohen-Macaulay module and $\dim U_M(0) \leq d - 2$.

In [V2] Vasconcelos asked whether, for any two minimal reductions J_1, J_2 of an \mathfrak{m} -primary ideal I , $e_1(J_1, M) = e_1(J_2, M)$? As an application of Corollary 4.9 we get an answer to this question when $M/U_M(0)$ is a Cohen-Macaulay module.

Corollary 4.11. *Let I be an \mathfrak{m} -primary ideal of R . Assume that R is a homomorphic image of a Cohen-Macaulay local ring and $M/U_M(0)$ a Cohen-Macaulay R -module. Then there exists a constant c such that $e_1(J, M) = c$ for all minimal reductions J of I .*

Proof. Let J be reduction of ideal I . Since $M/U_M(0)$ is Cohen-Macaulay, we get by Corollary 4.9 and Proposition 3.2 that

$$e_1(J, M) = -\text{arith-deg}_{d-1}(J, M) = \begin{cases} -e_0(J, U_M(0)) & \text{if } \dim U_M(0) = d-1, \\ 0 & \text{if } \dim U_M(0) < d-1. \end{cases}$$

The conclusion follows from a result of D. G. Northcott and D. Rees [NR], which says that $e_0(J, U_M(0)) = e_0(I, U_M(0))$ for all reduction ideals J of I . \square

It should be mentioned here that we do not need the assumptions that R is a homomorphic image of a Cohen-Macaulay ring and the parameter ideal \mathfrak{q} is distinguished in Theorem 4.2 for the case $\dim M \leq 2$. However, these hypotheses are essential in Theorem 4.5. So we close this paper with the following two examples which show that the assumptions that R is a homomorphic image of a Cohen-Macaulay ring and the parameter ideal \mathfrak{q} is distinguished in Theorem 4.5, can not omit when $\dim M \geq 3$.

Example 4.12. let $k[[X, Y, Z, W]]$ be the formal power series ring over a field k . We consider the local ring $S = k[[X, Y, Z, W]]/I$, where $I = (X) \cap (Y, Z, W)$. Then $\dim S = 3$ and $\mathcal{D} : S = D_0 \supset (X)/I = D_1 \supset D_2 = 0$ is the dimension filtration of S . By Lemma 3.4 we get that $e_1(Q, S) = e_1(Q, S/D_1) = 0$ for every parameter ideal Q of S . On the other hand, there exists by Nagata [N] a Noetherian local integral domain (R, \mathfrak{m}) so that $\hat{R} = S$, where \hat{R} is the \mathfrak{m} -adic completion of R . Let \mathfrak{q} be an arbitrary parameter ideal of R . Since R is a domain, \mathfrak{q} is distinguished. Moreover, since

$$e_1(\mathfrak{q}, R) = e_1(\mathfrak{q}S, S) = 0 = -\text{arith-deg}_2(\mathfrak{q}, R),$$

R satisfies the condition (4) of Theorem 4.5. But R is not a sequentially Cohen-Macaulay domain, as it is not Cohen-Macaulay.

Example 4.13. Let $R = k[[X, Y, Z, W]]$ be the formal power series ring over a field k . We look at the R -module

$$M = (k[[X, Y, Z, W]]/(X, Y) \cap (Z, W)) \bigoplus k[[X, Y, Z]]$$

Set $D_1 = k[[X, Y, Z, W]]/(X, Y) \cap (Z, W)$. Then $M \supset D_1 \supset 0$ is the dimension filtration of M and $\Lambda(M) = \{3; 2\}$. Moreover, D_1 is a Buchsbaum module, $\text{depth } M = \text{depth } D_1 = 1$ and so that M is not sequentially Cohen-Macaulay. We put $U = X - Z$, $V = Y - W$ and $Q = (U, V, X)$. Since M/D_1 is Cohen-Macaulay, $e_i(Q, M/D_1) = 0$ for all $i = 1, 2, 3$. Therefore by Lemma 3.4 and Proposition 3.2 we have

$$e_1(Q, M) = -e_0(Q, D_1) = -\text{arith-deg}_2(Q, M), \text{ and}$$

$$e_2(Q, M) = -e_1(Q, D_1) = 0 = \text{arith-deg}_1(Q, M).$$

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